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The return to isotropy of homogeneous turbulence

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The return to isotropy of homogeneous turbulence without mean velocity gradients is attacked by considering changes to be slow relative to turbulence time scales. This single assumption permits the problem to be cast as one of finding the form of three invariant functions. Examination of limiting behaviour for large Reynolds number and small anisotropy, as well as small Reynolds number and arbitrary anisotropy, places restrictions on the form of the functions. Realizability conditions (requiring that energies be non-negative) reduce the problem to two functions subject to further restrictions. A convenient interpolation form is found for the functions, satisfying all the restrictions, and it is shown that predictions based on this are in excellent agreement with all available data.

1. Introduction

The question of whether, and how fast, anisotropic (homogeneous, isothermal) turbulence without mean velocity gradients returns to isotropy is of fairly long standing. Rotta (1951) suggested an approximate form for the pressure-gradient/velocity correlation which is responsible for the intercomponent energy interchange; this form being proportional to the energy deficit in the component, a return to isotropy with a single time constant is predicted. Although there is no analytical proof that this is so, there is folk wisdom in the turbulence community that, in the absence of external agencies, anisotropy should decrease to avoid a second-law violation. The first measurements of this were made by Uberoi (1956, 1957), followed by Mills & Corrsin (1959). These do indeed show a return to isotropy, although the downstream distance is painfully short and the indications near the ends of the ducts are (if end effects may be excluded, which is not clear) that the return is weak. Admittedly, in Uberoi (1956) and Mills & Corrsin (1959) it was not the major purpose to investigate the return to isotropy following the contraction, but rather to examine the effect of the contraction itself.

We intend to approach the question in a way reminiscent of that in which the school of rational mechanics approached the question of constitutive relations for non-Newtonian media. Eschewing detailed knowledge of the mechanisms responsible for the stress, rational mechanics attempted to determine the mathematical constraints to which constitutive relations were subject, and from these to delineate the possible forms of these relations. When a relation had been reduced to a small number of

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invariant functions, a combination of limiting cases and experiment could be used to determine these functions, establishing once and for all the behaviour in a range of possible situations. Although we know more about the detailed mechanisms of turbulence than we know about the statistical mechanics of complex media, it is evident that we do not yet know enough to make detailed predictions from first principles of such behaviour as the return to isotropy; there is therefore some justification for taking a similar approach to turbulence. We shall finish up with a rational construction which, although it may be termed semi-empirical, behaves like turbulence in a range of circumstances. The concept of viscosity, also, may be shown rationally to describe the behaviour of certain media in a range of circumstances, and since the value of the viscosity in a particular medium must be determined from experiment, it may be said to be semi-empirical.

There is, in fact, considerable similarity between the return-to-isotropy problem and the behaviour of non-Newtonian media. In a turbulent flow at high Reynolds number, the only important stress is the Reynolds stress $-\overline{u_i u_j}$. In the experiments referred to, initially approximately isotropic turbulence is subjected to an axisymmetric contraction to produce the anisotropy, which is then allowed to decay. Measurements are taken in the principal axes of the Reynolds-stress tensor. Hence the equivalent problem in a non-Newtonian medium would be the subjection of material initially in a state of isotropic stress to an axisymmetric distortion which produces unequal normal stresses which are then allowed to decay to equality. It is, of course, dangerous to draw simplistic analogies: turbulence cannot be regarded as a non-Newtonian medium for a number of reasons. Chief among these are the fact that it does not satisfy the principles of material indifference or determinism (Lumley 1970*a*), that there are regions near boundaries in time and space where local states are not unique functionals of flow variables (Lumley 1970*a*) and that the turbulent energy, which is responsible for the transport phenomena, is maintained by the external energy input. The latter makes the mechanics of turbulence evocative of (though in no way analogous to) the laminar mechanics of a gas at constant density and very low temperature in the sense that molecular momentum transport will be substantial only where the temperature is substantial owing to shear or some other energy input. Nevertheless, in homogeneous turbulent flows many of the techniques and concepts of continuum mechanics can be applied, with gratifying results.

2. Formulation of the problem

We may write the equations of a decaying, anisotropic, homogeneous, isothermal turbulence field without mean velocity as follows:

$$\left.\begin{aligned} \dot{\overline{u_i u_j}} &= -(\overline{p_{,i} u_j} + \overline{p_{,j} u_i})/\rho - 2\nu \overline{u_{i,k} u_{j,k}} \\ &= [-(\overline{p_{,i} u_j} + \overline{p_{,j} u_i})/\rho - 2\nu \overline{u_{i,k} u_{j,k}} + \tfrac{2}{3}\epsilon\delta_{ij}] - \tfrac{2}{3}\epsilon\delta_{ij} \\ &= -\epsilon\phi_{ij} - \tfrac{2}{3}\epsilon\delta_{ij}, \\ \dot{\epsilon} &= -2\nu \overline{u_{i,k} u_{i,j} u_{j,k}} - 2\nu^2 \overline{u_{i,kj} u_{i,kj}} = -\psi\epsilon^2/q^2. \end{aligned}\right\} \tag{1}$$

The final term in the first line is the dissipation of $\overline{u_i u_j}$ stuff to heat. This term is observed (Monin & Yaglom 1975, p. 453) to become more isotropic with increasing

Reynolds number, in agreement with Kolmogorov's (1941) hypothesis of local isotropy. We shall be continually using the state of infinite Reynolds number as a reference state, and it will be convenient for us to add and subtract the form of this term at infinite Reynolds number as we have done in the second line of (1), incorporating the difference (which vanishes at infinite Reynolds number) in the dimensionless term on the third line. ϵ is the total dissipation $\nu\overline{u_{i,k}u_{i,k}}$ of mechanical energy (per unit mass) into heat, while $q^2 = \overline{u_i u_i}$. The equation for the dissipation, the fourth line of (1), has been treated elsewhere (Lumley & Khajeh-Nouri 1972); we shall be able to do little to improve that treatment here owing principally to a lack of data. It is primarily to ϕ_{ij} that we wish to give our attention here.

Now, we may regard (1) as equations for ϕ_{ij} and ψ. That is, if we knew the history of $\overline{u_i u_j}$ and ϵ and the value of the viscosity, we could solve for ϕ_{ij} and ψ. Hence they may be written as functionals as

$$\phi_{ij} = \phi_{ij}\{\overline{u_i u_j}, \epsilon, \nu\}, \quad \psi = \psi\{\overline{u_i u_j}, \epsilon, \nu\}, \tag{2}$$

where the range of the functionals is over the entire past. We have assumed that there is no explicit dependence on time. This is equivalent to assuming that the state of the turbulence at an arbitrary instant is completely determined by the history of the motion relative to that instant; i.e. that the nature of turbulence is the same from day to day, or in the words of Francis Clauser, 'that there is no clock in the (black) box'. If we presume that turbulence has a fading memory and introduce the concept of a retarded history (Coleman & Noll 1961) we may carry out an expansion of these functionals in the derivatives of the arguments (and their products) at the current time (Volterra 1959, p. 25). If changes in the state of the system are sufficiently slow relative to the memory time of the turbulence (in decaying turbulence the ratio of these scales is between $\frac{1}{3}$ and $\frac{2}{3}$ depending on definitions), we may neglect higher derivatives and terminate the series at some point. It is easy to see that, if we keep derivatives no higher than the first, these may be incorporated in the time derivative on the left, to give the same equations as (1), where now ϕ_{ij} and ψ are *functions* (not functionals) of the present state. Although the time-scale ratio in real turbulence may not be regarded as small, in fact the approximations obtained from this assumption will be shown to work remarkably well. This is equally true of the corresponding length-scale assumption (Lumley, Zeman & Seiss 1977) when properly applied.

The expressions in (2) being now (dimensionless) functions of their arguments, we may use dimensional analysis to reduce their form. Each of these functions has eight arguments, containing the dimensions of length and time only. Hence six independent dimensionless quantities can be formed. We choose to do this in a way which isolates the property of anisotropy from the other properties, and form the tensor

$$b_{ij} = \overline{u_i u_j}/q^2 - \tfrac{1}{3}\delta_{ij}, \tag{3}$$

which is dimensionless, has zero trace, and vanishes identically if the turbulence is isotropic. Hence it has five independent components. The sixth is formed from the trace of the Reynolds stress (the energy) and the dissipation and viscosity:

$$R_l = q^4/9\epsilon\nu, \tag{4}$$

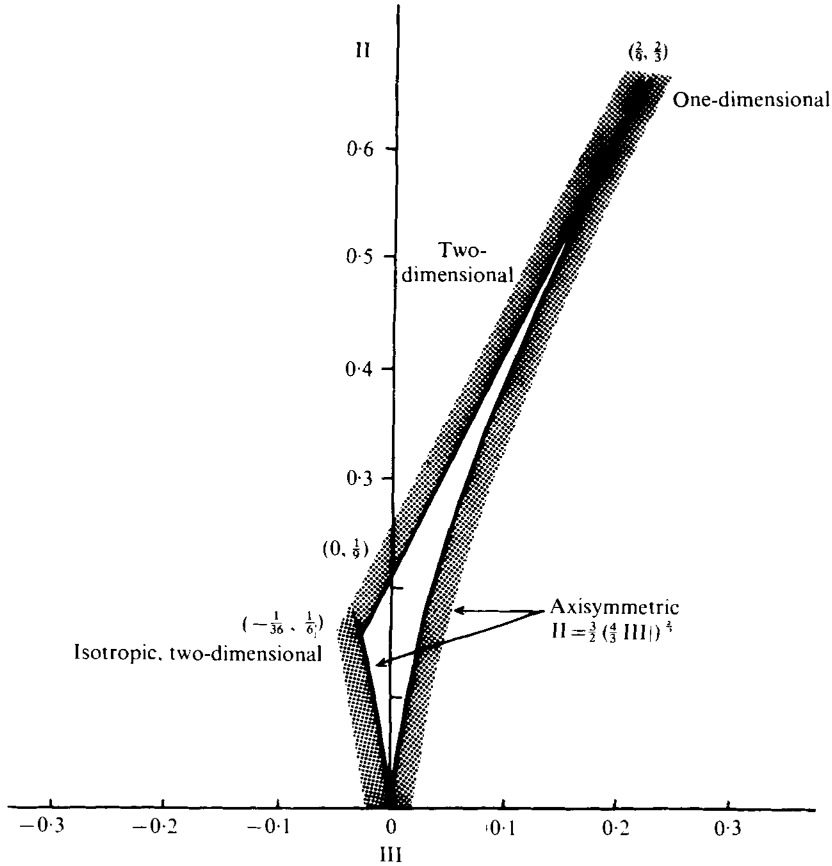

FIGURE 1. Plot of limiting values of the second and third invariants for turbulence, which must exist within the area delimited by the shading.

a Reynolds number. The factor of nine reduces this to the classical definition of R_l when $\epsilon = u^3/l$ and $q^2 = 3u^2$ are substituted. Thus

$$\left.\begin{aligned} \phi_{ij}(\overline{u_i u_j}, \epsilon, \nu) &= \phi_{ij}(\mathbf{b}, q^2, \epsilon, \nu) = \phi_{ij}(\mathbf{b}, R_l), \\ \psi(\overline{u_i u_j}, \epsilon, \nu) &= \psi(\mathbf{b}, q^2, \epsilon, \nu) = \psi(\mathbf{b}, R_l). \end{aligned}\right\} \tag{5}$$

We may now apply invariant theory to the functions in (5) (see Lumley 1970*b*, p. 179). That is to say, in writing (2) or (5), we have explicitly indicated all arguments; nothing else appears in (1), and the only other way in which an argument could be introduced would be through the boundary conditions. Here, however, there are none, since the field is homogeneous. Thus there are no suppressed, or hidden, arguments in (2) or (5); the functions in (5) must therefore be isotropic functions of the indicated arguments. This does not mean that the value of the function for a particular value of the argument is isotropic, but that the relationship is isotropic; i.e. that any anisotropy present in the value of the function is induced by anisotropy of the argument, and not by the presence of another suppressed argument, such as the direction of a magnetic field or the orientation of a nearby boundary. The functional relationship must thus be invariant under the rotation group in three dimensions, and this restricts its form. We find

$$\left.\begin{aligned}\phi_{ij}(\mathbf{b}, R_l) &= b_{ij}\beta + \gamma(b^2_{ij} - \tfrac{1}{3}\delta_{ij}\,\mathrm{II}),\\ \beta = \beta(\mathrm{II},\mathrm{III},R_l), \quad \gamma &= \gamma(\mathrm{II},\mathrm{III},R_l), \quad \psi = \psi(\mathrm{II},\mathrm{III},R_l),\\ \mathrm{II} = b_{ij}b_{ij}, &\quad \mathrm{III} = b_{ij}b_{jk}b_{ki}\end{aligned}\right\} \tag{6}$$

(since the trace of ϕ_{ij} must vanish). We have now reduced the problem to the determination of the form of three invariant functions of the independent invariants of the anisotropy tensor and the Reynolds number.

It is interesting to examine the state of the turbulence as specified by the two invariants II and III (our notation is slightly unconventional; ordinarily II and III are used for the principal invariants, but these differ by only a numerical factor from our definitions). In figure 1 we have sketched the possible values of II and III, with the corresponding states of the turbulence; all homogeneous turbulence must be found within the shaded area. The right and left curved boundaries correspond to axisymmetric turbulence, the right to that in which the axial component is larger than the other two and the left to that in which it is smaller. The elbow on the left corresponds to the vanishing of the axial component, leaving isotropic two-dimensional turbulence. Along the diagonal straight line the turbulence is two-dimensional, until at the summit one of the components has vanished, leaving one-dimensional turbulence; this point may also be reached from the right-hand axisymmetric state, corresponding to the vanishing of both transverse components. The narrow range of values of III suggests an approximation in which III is taken as a function of II, lying along the centre of the region, although no such approximation has been used here.

3. Asymptotic behaviour of the invariant functions

We shall now attempt to extract from various asymptotic states information about the behaviour of the invariant functions β and γ. We shall delay treatment of the invariant function ψ until a later section, as the considerations are somewhat different and less conclusive.

In the final period of decay (Batchelor 1956, p. 92) the momentum equation may be written as

$$\overline{\dot{u_i u_j}} = -2\nu\overline{u_{i,k}u_{j,k}} = -\epsilon(2b_{ij} + \tfrac{2}{3}\delta_{ij}), \quad R_l \to 0. \tag{7}$$

This state corresponds to very small Reynolds number, and the consequent negligibility of all inertial terms. Hence there is no longer any interchange between the components, and each decays at the same rate. Equation (7) implies that

$$\phi_{ij} \to 2b_{ij}, \quad R_l \to 0. \tag{8}$$

In the particular case of axisymmetric turbulence (which we shall use frequently, since all the experiments have been carried out on such turbulence) we have

$$\mathbf{b} = \begin{pmatrix} b & 0 & 0 \\ 0 & -\tfrac{1}{2}b & 0 \\ 0 & 0 & -\tfrac{1}{2}b \end{pmatrix} \tag{9}$$

and in the final period we have

$$\phi_{11}/b \to 2, \quad R_l \to 0. \tag{10}$$

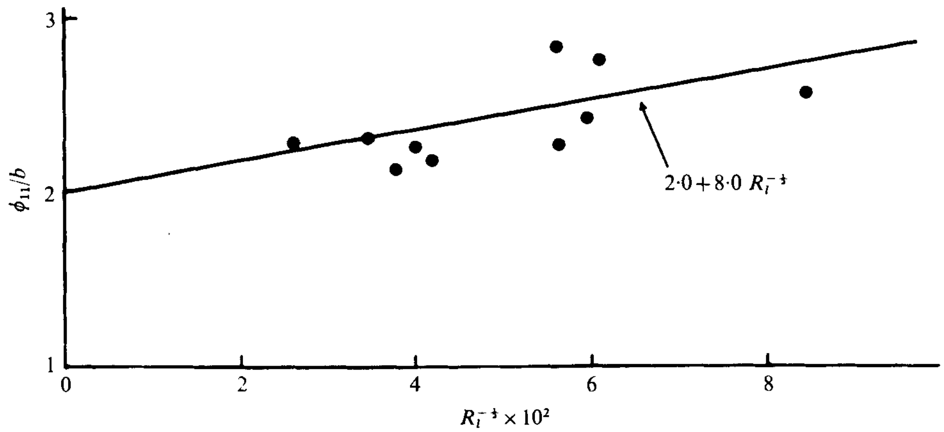

FIGURE 2. Return to isotropy of slightly anisotropic turbulence, as a function of Reynolds number, from the data of Comte-Bellot & Corrsin (1966); II → 0.

The value of 2 will arise frequently; it corresponds to zero return to isotropy, in the sense that each component maintains a constant ratio to the energy. It appears in (10) because of our inclusion of the anisotropic part of the dissipation in ϕ_{ij}.

We may now consider the data of Comte-Bellot & Corrsin (1966), who carried out extensive experiments on very slightly anisotropic turbulence at various (moderately large) Reynolds numbers. The ratio ϕ_{11}/b may be calculated from their ten experiments without a secondary contraction. At large Reynolds numbers, we may expect that the dependence on Reynolds number will be through the ratio of time scales, proportional to $R_l^{-\frac{1}{2}}$, in accord with the reasoning in Tennekes & Lumley (1972, p. 89). If we plot the ratio ϕ_{11}/b against $R_l^{-\frac{1}{2}}$ as in figure 2, a least-squares fit gives $1 \cdot 960 + 8 \cdot 854\ R_l^{-\frac{1}{2}}$, with a correlation coefficient of 0·61. Values of ϕ_{11}/b less than 2 correspond to an increase of anisotropy in the absence of external agencies, and although there is no proof, this sounds like a second-law violation. Since the accuracy of the data is probably not better than 5 %, we may safely take the line $2 \cdot 0 + 8 \cdot 0\ R_l^{-\frac{1}{2}}$ obtained by rotating the regression line about the centroid to pass through 2·0. Thus

$$\phi_{11}/b \to 2 \cdot 0 + 8 \cdot 0/R_l^{\frac{1}{2}}, \quad R_l \to \infty, \quad \mathrm{II} \to 0. \tag{11}$$

This suggests that there is no linear return to isotropy at infinite Reynolds number; i.e. that when viscosity is negligible, the return to isotropy is an entirely nonlinear effect and vanishes as the anisotropy vanishes. This may help to explain the evidently very weak return to isotropy for small anisotropies.

We may now undertake an analytical examination of the equations for large Reynolds number and small anisotropy, using the result (11). We may write the equation for the anisotropy tensor $\mathbf{b}$ as

$$q^2 \dot{b}_{ij}/\epsilon = -[(\beta - 2)\, b_{ij} + \gamma(b_{ij}^2 - \tfrac{1}{3}\delta_{ij}\mathrm{II})] = -(\phi_{ij} - 2b_{ij}), \tag{12}$$

from which we may immediately obtain the equations for the invariants II and III. It is most convenient to use a natural time defined by

$$dt(\epsilon/q^2) = d\tau, \quad \tau - \tau_0 = -\tfrac{1}{2}\ln(q^2/q_0^2), \tag{13}$$

which is monotone, since the energy in these flows decreases continually. In terms of this time, we have

$$d\mathrm{II}/d\tau = -2[(\beta-2)\,\mathrm{II}+\gamma\mathrm{III}], \quad d\mathrm{III}/d\tau = -3[(\beta-2)\,\mathrm{III}+\tfrac{1}{6}\gamma\mathrm{II}^2]. \tag{14}$$

We may now consider that the turbulence is nearly isotropic, so that $\mathrm{II}\to 0$ and $\mathrm{III}\to 0$, and that the Reynolds number becomes very large. Then, if we presume that (11) holds and that β and γ are analytic in the invariants, we may expand

$$\beta-2 = \beta_1\mathrm{II}+\beta_2\mathrm{III}+\ldots, \quad \gamma = \gamma_0+\gamma_1\mathrm{II}+\gamma_2\mathrm{III}+\ldots. \tag{15}$$

Keeping only the lowest-order terms, we find that $d\mathrm{II}/d\tau = -2\gamma_0\mathrm{III}$, so that any value of γ_0 is inconsistent. That is, for vanishing anisotropy, no matter what sign is chosen for γ_0 there is always a sign of III which will permit the anisotropy to increase. This is again a thermodynamic sort of consideration, and must be taken on faith. In a similar way, we find that β_1 must be positive. Thus, for the final relaxation of anisotropy at infinite Reynolds number we have

$$d\mathrm{III}/d\tau \cong -3\beta_1\mathrm{II}\,\mathrm{III}, \quad d\mathrm{II}/d\tau \cong -2\beta_1\mathrm{II}^2 \tag{16}$$

(so that II and III both decrease, since II is non-negative), which has the solution

$$\mathrm{II} \propto 1/\ln t. \tag{17}$$

This slow behaviour is quite consistent with the observed weak return to isotropy.

In the axisymmetric case, we have

$$\mathrm{II} = \tfrac{3}{2}b^2, \quad \mathrm{III} = \tfrac{3}{4}b^3 \tag{18}$$

and we can write

$$\phi_{11}/b \cong 2+\beta_1\mathrm{II}+O(b^3). \tag{19}$$

This form has been suggested before (Lumley & Khajeh-Nouri 1972), and has been objected to on the grounds that the data of Comte-Bellot & Corrsin (1966) did not support it (Reynolds 1976). The objection was well founded, since a least-squares analysis of the remaining variance in the Comte-Bellot & Corrsin (1966) data using a form such as (19) gives a correlation coefficient of only 0·1. We now see, however, that the term is logically essential, and we shall find that the coefficient is substantial. Schumann & Patterson (1977) fit a form such as (19) to the results of numerical simulations, and obtain a negative value for β_1. We shall show later how this comes about (because the rate of return to isotropy at first increases, but then decreases for large anisotropy); we can see here that this is logically impossible, however.

4. Realizability

Schumann (1977) has pointed out that, if one component of the energy (in principal axes) vanishes, the time derivative of that component must vanish also, in order to avoid subsequent negative values of that component. If we cast our equation in principal axes, indicating this by the use of Greek subscripts, eigenvalues being indicated by a single subscript in parentheses, we have

$$-\epsilon\phi_{(\alpha)} = -\overline{2p_{,\alpha}u_\alpha}/\rho - \overline{2\nu u_{\alpha,k}u_{\alpha,k}}+\tfrac{2}{3}\epsilon. \tag{20}$$

Now, if $u_\alpha \to 0$ (which corresponds to $b_{(\alpha)} \to -\frac{1}{3}$), then we must have $\phi_{(\alpha)} \to -\frac{2}{3}$ (because of the inclusion of the anisotropic part of the dissipation term in ϕ_{ij}). Thus we must have $\phi_{(\alpha)}/b_{(\alpha)} \to 2$ as $b_{(\alpha)} \to -\frac{1}{3}$ (which in the axisymmetric case with III < 0 corresponds to the vanishing of the axial component, for a value of II $= \frac{1}{6}$).

The form (6) is completely general. For our purposes it will be sufficient to assume a slightly less general form, namely that ϕ_{ij} is a polynomial of arbitrary order in b_{ij} (i.e. of the form $a_0 \mathbf{b}^0 + a_1 \mathbf{b}^1 + a_2 \mathbf{b}^2 + \dots$, where $b^n_{ij} = b_{ip} b_{qp} \dots b_{qj}$ with n factors, and the a_i are constants), minus the trace of this polynomial. This introduces relations among the coefficients of β and γ, but these do not appear to be unduly restrictive, and occur only at an order higher than that which we have considered. For example, β_0, γ_0 and β_1 are unrestricted, but $\beta_2 = -\gamma_1$ is required. Functions such as exp $\mathbf{b}$ and ln $\mathbf{b}$, as usually defined, are of this form. With this assumption, we may write in principal axes

$$\phi_{ij} = \tfrac{1}{3}\begin{pmatrix} 2f(b_{(1)}) - f(b_{(2)}) - f(b_{(3)}) & 0 & 0 \\ 0 & 2f(b_{(2)}) - f(b_{(1)}) - f(b_{(3)}) & 0 \\ 0 & 0 & 2f(b_{(3)}) - f(b_{(1)}) - f(b_{(2)}) \end{pmatrix}, \quad (21)$$

where $f(x)$ is the polynomial function. The realizability condition now requires that, if any eigenvalue of b_{ij} take the value $-\frac{1}{3}$, then the corresponding eigenvalue of ϕ_{ii} must take the value $-\frac{2}{3}$, regardless of the value of the other eigenvalues:

$$[2f(-\tfrac{1}{3}) - f(b_{(2)}) - \tfrac{1}{3}f(\tfrac{1}{3} - b_{(2)})] = -\tfrac{2}{3}. \quad (22)$$

This is a functional equation for $f(x)$, which may readily be solved to give

$$f(x) = g(x - \tfrac{1}{6}), \quad g(x) = -g(-x), \quad g(\tfrac{1}{2}) = +1. \quad (23)$$

The conditions we have previously derived may be expressed as conditions on $g(x)$:

$$\left.\begin{aligned} \gamma_0 = 0, \quad & \text{so that} \quad g''(\tfrac{1}{6}) = 0, \\ \phi_{11}/b \to 2, \quad R_l \to \infty, \quad & \text{so that} \quad g'(\tfrac{1}{6}) = 2. \end{aligned}\right\} \quad (24)$$

These various conditions serve to determine the form of $f(x)$ quite precisely. It is convenient to write $g(x) = 2x + h(x)$, from which we find that $h(x) = -h(-x)$ and $h(\frac{1}{2}) = 0$ and conditions (24) give $h'(\frac{1}{6}) = 0$ and $h''(\frac{1}{6}) = 0$. This suggests using a Fourier series in sin $2\pi nx$ to satisfy the first two requirements. Requirements (24) or the equivalent restrictions on $h(x)$ are infinite Reynolds number requirements; we shall have to modify them later for finite Reynolds number. For the moment, let us consider the case of infinite Reynolds number. Keeping three terms, which will prove to be more than sufficient for our purposes, we find

$$g(x) = 2x + a(4 \sin 2\pi x - \sin 4\pi x + \sin 6\pi x), \quad (25)$$

where a is an undetermined constant. We cannot determine a from the data directly, since the data are all at quite low Reynolds number. We must instead devise a suitable interpolation formula for finite Reynolds number.

In the axisymmetric case, we have relation (11). We can write for the axisymmetric case

$$\phi_{11}/b = 2[g(\tfrac{1}{6} + \tfrac{1}{2}b) - g(\tfrac{1}{6} - b)]/3b \quad (26)$$

and for small b this becomes

$$\phi_{11}/b = g'(\tfrac{1}{6}) + \dots . \quad (27)$$

Hence evidently

$$h'(\tfrac{1}{6}) = 8{\cdot}0/R_l^{\frac{1}{2}} + \ldots \tag{28}$$

for large but finite Reynolds numbers. We also have the requirement (10) that h must vanish as $R_l \to 0$. The most convenient way of meeting this requirement is to write

$$g(x) = 2x + F(R_l)\,h(x), \tag{29}$$

where $F(\infty) = 1$ and $F(0) = 0$. If we use large Reynolds number forms such as (28) for all Reynolds numbers in expression (29), $h(x)$ will increase without bound as R_l goes to zero; hence F must go to zero sufficiently rapidly to dominate this term. In the range of $R_l^{-\frac{1}{2}}$ of figure 2, it must differ negligibly from unity. We have found

$$F(R_l) = \exp(-DR_l^{-\frac{1}{2}}) \tag{30}$$

(where D is a coefficient to be determined by fitting the data) to be quite satisfactory. There is no theoretical justification for this form, but it collapses the data over a range of R_l of approximately one order of magnitude.

One final point must be examined. We have shown that γ_0 must vanish at infinite Reynolds number, when β_0 is 2. At finite Reynolds number, however, we have (28), and the restriction on the value of γ_0 will be different. By examining the equations (14) for small values of II, in the axisymmetric case, we find for positive and negative values of III respectively

$$\left.\begin{aligned} \alpha \mathrm{II}^{\frac{1}{2}} 6^{\frac{1}{2}} &\geqslant -1, \quad 0 \leqslant \mathrm{II} \leqslant \tfrac{2}{3}, \\ \alpha \mathrm{II}^{\frac{1}{2}} 6^{\frac{1}{2}} &\leqslant +1, \quad 0 \leqslant \mathrm{II} \leqslant \tfrac{1}{6}, \quad \alpha = \gamma_0/6(\beta_0 - 2) \end{aligned}\right\} \tag{31}$$

from the requirement that III should decrease in absolute magnitude. The restriction for positive values of III is clearly always satisfied if α is positive. That for negative values will be satisfied for any positive value of α if II is small enough. This implies that γ_0 must vanish as β_0 approaches 2, which we knew, but that it must do so in such a way that α remains bounded. The easiest solution is to pick a constant value of α, although there is no formal justification for this. Although restrictions (31) are valid only for small values of II, the second restriction would be satisfied for all possible values of II if we restricted α to $0 < \alpha < 1$. Fortunately, the behaviour of the function is not terribly sensitive to the value of α, and the data are insufficient to distinguish between various values of α. We have picked a compromise value of $\alpha = \frac{1}{2}$. Thus we have

$$h''(\tfrac{1}{6}) = -2\gamma_0 = -12\alpha h'(\tfrac{1}{6}) = -6h'(\tfrac{1}{6}). \tag{32}$$

A similar discussion for II is unnecessary since from (11) we have

$$d\mathrm{II}/d\tau = -16\mathrm{II}/R_l^{\frac{1}{2}}$$

for small anisotropy and large R_l, so that II always decreases.

We find as a general form for $h(x)$, keeping terms no higher than $\sin 6\pi x$,

$$\begin{aligned} h(x) &= a(4\sin 2\pi x - \sin 4\pi x + \sin 6\pi x) \\ &\quad + (2h'(\tfrac{1}{6})/3\pi)\,[\alpha \times 3^{\frac{1}{2}}/\pi + 1)\sin 2\pi x + (\alpha \times 3^{\frac{1}{2}}/2\pi - \tfrac{1}{4})\sin 4\pi x]. \end{aligned} \tag{33}$$

Using (28)–(30) and the value $\alpha = \frac{1}{2}$, we have

$$\begin{aligned} g(x) &= 2x + \exp(-DR_l^{-\frac{1}{2}})\,[a(4\sin 2\pi x - \sin 4\pi x + \sin 6\pi x) \\ &\quad + 1{\cdot}70\,R_l^{-\frac{1}{2}}(1{\cdot}28\sin 2\pi x - 0{\cdot}112\sin 4\pi x)]. \end{aligned} \tag{34}$$

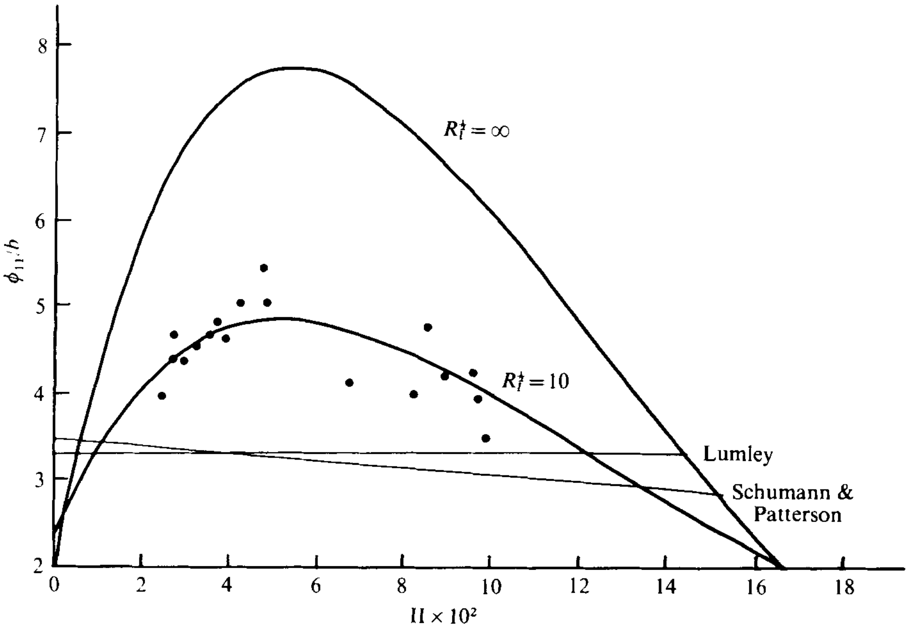

FIGURE 3. Form for axisymmetric (III < 0) flows of the return-to-isotropy function, with the data of Uberoi (1957) and Mills & Corrsin (1959) reduced to a common Reynolds number. The straight lines correspond to the forms assumed by Lumley (1975) and by Schumann & Patterson (1977).

Values of ϕ_{11}/b, $\mathrm{II} = \frac{3}{2}b^2$ and R_l were calculated from published data of Uberoi (1957) (using the three flows having the greatest streamwise extent, at three grid Reynolds numbers of 3710, 6150 and 12300) and Mills & Corrsin (1959), which gave in all 18 sets of points; the values of a and D were optimized to minimize the mean-square relative error at these points, which resulted in $D = 7{\cdot}77$ and $a = 0{\cdot}633$ with an r.m.s. error of 7 %.

The points calculated from the data were then reduced to a common Reynolds number of $R_l = 100$ by multiplying the observed value of ϕ_{11}/b by the ratio of ϕ_{11}/b calculated using (26) and (34) at $R_l = 100$ to that calculated at the observed Reynolds number. In figure 3 we show the form of ϕ_{11}/b predicted by (34) for an axisymmetric (negative III) flow for $R_l = 100$ and the data points reduced to this Reynolds number; we also show the infinite Reynolds number case (25). Note that the location of the peak (and indeed the general form of the curve) is not adjustable; effectively, only the amplitude has been adjusted empirically. Note that the slope of the curve for infinite Reynolds number is substantial near the origin, being roughly $2 + 132\mathrm{II} + \ldots$; because of the decrease in the return rate with increasing anisotropy, however, an approximation of this form would very much overpredict. It is easy to see how, on the basis of the data points, a larger intercept and smaller slope would appear reasonable. If only flows with substantial anisotropy were examined, a negative slope would be quite acceptable, corresponding to the back portion of the curve. This probably explains the results of Schumann & Patterson (1977). The straight line with negative slope corresponds to their results, regressed linearly to $R_l = 100$, while the horizontal straight line is the Rotta model used by Lumley (1975).

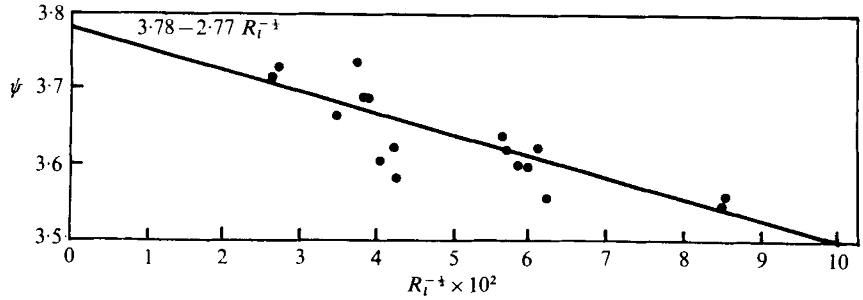

FIGURE 4. Dimensionless decay of dissipation as a function of Reynolds number, from the data of Comte-Bellot & Corrsin (1966).

In a general flow, in order to use these ideas we should have to find the eigenvalues and eigenvectors of the anisotropy tensor, and then form

$$\left.\begin{aligned} \phi_{ij} = \tilde{\phi}_{ij} - \tfrac{1}{3}\delta_{ij}\tilde{\phi}_{pp}, \quad \tilde{\phi}_{ij} = \sum_{\alpha=1}^{3} g(b_{(\alpha)} - \tfrac{1}{6})\, X_i^{(\alpha)} X_j^{(\alpha)}, \\ b_{ij} X_j^{(\alpha)} = b_{(\alpha)} X_i^{(\alpha)}, \end{aligned}\right\} \tag{35}$$

where $g(x)$ is given by (34). This is tedious, but has the advantage, in addition to behaving properly with the Reynolds number and anisotropy, of guaranteeing non-negative energies.†

5. The dissipation equation

As far as the form of ψ is concerned, we are in a much less satisfactory position. Again, the data of Comte-Bellot & Corrsin (1966) provide the asymptotic behaviour for large Reynolds number and very small anisotropy, giving

$$\psi = 3{\cdot}78 - 2{\cdot}77\, R_l^{-\frac{1}{2}} - 9{\cdot}09\,\mathrm{II} + \dots \tag{36}$$

with a correlation coefficient of 0·8 for the Reynolds number variation and a correlation coefficient of 0·33 for the variation with anisotropy (see figure 4). Hence the former is quite reliable, while the latter is uncertain. In the final period of decay, for very small Reynolds number and all anisotropies we find $\psi = \frac{14}{5} = 2{\cdot}80$.

The case of one-dimensional turbulence can be solved exactly, since there is no nonlinear interaction; the equations are thus the same as those for the final period of decay. Because of the one-dimensional nature of the spectrum, however, we get $\psi = 3$, instead of $\frac{14}{5}$. When we consider the result for the final period, however, and let the spectrum become increasingly one-dimensional, we find $\psi = \frac{14}{5}$ right up to the point of one-dimensionality. This is evidently a singular limit; for any intensity of turbulence, we can imagine turbulence sufficiently one-dimensional so that the nonlinear terms may be neglected, and we shall get the result corresponding to the final period of decay. Thus we shall take $\psi = \frac{14}{5}$ for $\mathrm{II} = \frac{2}{3}$ and an arbitrary Reynolds number, recognizing that this is a limit from values of $\mathrm{II} < \frac{2}{3}$.

Realizability is already satisfied, since the expression is multiplied by ϵ^2, so that $\dot{\epsilon}$ will vanish when ϵ vanishes.

† The guarantee is absolute only if the directions of the principal axes do not change with time.

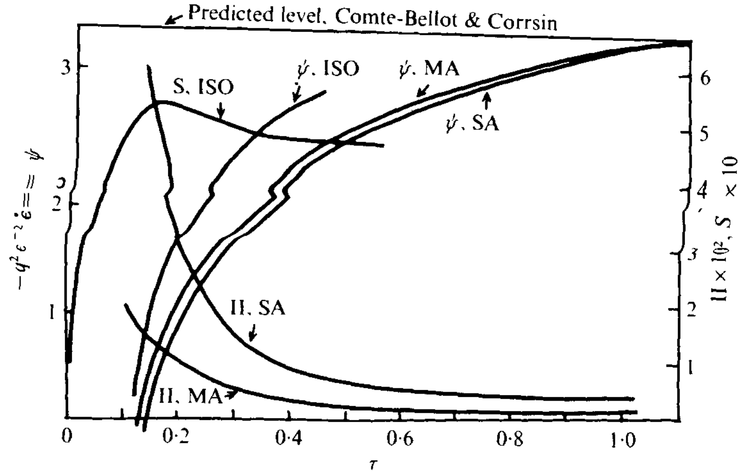

FIGURE 5. Various quantities computed from the exact simulations of isotropic and axisymmetric turbulence of Orszag & Patterson (1972) and Schumann & Herring (1976). The symbols are identified in the text.

The information we need is that which should be supplied by experiment on the behaviour of ψ at moderate anisotropy and moderate Reynolds number. Unfortunately, in this respect the experiments are very unsatisfactory. The values of ψ are essentially the second derivative of the energy, and it is nearly impossible to obtain this from scattered data. About all that can be concluded from the data of Uberoi (1957) is that ψ is positive. The data of Mills & Corrsin (1959) do not agree even on this, suggesting the opposite curvature. Fortunately, the following simple test can be applied to grid turbulence data.

For decaying isotropic turbulence, at any downstream position during the initial period of decay the distance from its virtual origin (its effective age) is given by

$$n\tilde{q}^2/2\tilde{\epsilon} = \tilde{x}_0, \tag{37}$$

where all variables are non-dimensionalized by the mean velocity and the mesh size and n is the power in the decay law. Practically speaking, there is little difference between the distance from the grid and the distance from the virtual origin, usually less than a few mesh lengths (Comte-Bellot & Corrsin 1966). In going through a mild contraction, this apparent age should increase somewhat, since the energy will be increased, and the dissipation reduced, by the energy input during the contraction. In fact, just after the contraction the three flows of Uberoi have values of $\tilde{x}_0$ which correspond roughly to the streamwise position at which the contraction began. The flow of Mills & Corrsin, however, has a value of $\tilde{x}_0$ just after the contraction corresponding to a streamwise distance approximately three times that at which the contraction began, in fact half-way to the end of the tunnel. Since the contraction ratio is the same as that for the Uberoi flows, this suggests that something is seriously wrong with the data. The difficulty does not seem to extend to the values of ϕ_{11}/b, which agree well with the data of Uberoi; a possible explanation is that the latter quantity is determined from b or equivalently II, which is independent of the calibration, whereas the energy itself will be influenced by calibration errors.

A serious effort was made to obtain information from the exact simulations of Orszag

& Patterson (1972) and Schumann & Herring (1976), who have calculated respectively isotropic and axisymmetric homogeneous turbulence without mean velocity gradients. From numerical results kindly supplied by the authors, values of ψ were calculated, and these are plotted in figure 5 against $\tau = -\frac{1}{2}\ln q^2$ for Orszag & Patterson's isotropic flow and the weakly anisotropic (WA) and strongly anisotropic (SA) flows of Schumann & Herring. It may be seen that the calculated values of ψ are tending towards the value suggested by the experiments of Comte-Bellot & Corrsin (1966), but slowly. For comparison, the values of the skewness for isotropic flow are indicated on the same plot, and it may be seen that this is approaching its asymptotic value much more rapidly. The values of ψ do not begin to rise until the skewness has essentially reached its final value, indicating that spectral transfer has been established. Also indicated on the same plot are the values of II for the two anisotropic flows; it may be seen that the anisotropy is essentially gone by the time ψ is near the values appropriate to real turbulence.

Orszag & Herring (private communication) have indicated that the slow approach of ψ to its asymptotic value is probably due to the use of an initial spectrum of the form $k^4 \exp(-k^2)$, which takes a long time to evolve to a self-preserving form. They indicate that a spectrum of the form $k\exp(-k)$ evolves much more rapidly, although differencing errors appear to be introduced by this spectrum, requiring a $(64)^3$ code rather than a $(32)^3$ code. This explanation is in agreement with what is known of spectral dynamics: ψ is essentially the imbalance between vortex stretching by the fluctuating strain rate and molecular transport (Tennekes & Lumley 1972, p. 91), and hence is controlled by the large wavenumber end of the spectrum. It should take about one decay time ($\tau = 1$) to set up an equilibrium high wavenumber spectrum, since this is the time required for energy to traverse the spectrum. It is hoped that at some time in the not too distant future more realistic simulations will be available which will be of some use in determining the form of ψ.

We also tried to use these calculations to obtain values of ϕ_{11}/b. If values before the skewness peak are excluded, only the strong anisotropy case provides interesting values of II, and these all lie below 3×10^{-2}; the corresponding values of ϕ_{11}/b follow the shape of the curve in figure 3, but lie between the $R_l = 100$ and $R_l = \infty$ curves and hence are too high, since R_l for these calculations is about 20. Evidently, the setting-up of an equilibrium spectrum, which increases the decay substantially (giving low values of ψ), also increases the return to isotropy.

Thus we are at a loss to determine the behaviour of ψ more precisely than can be done from (36) and the various asymptotic cases. The not very reliable indication of (36) that small anisotropy decreases the rate of destruction of ϵ, resulting in higher ϵ levels and hence faster decay, makes physical sense and works well in other flows (Zeman 1975). What should happen at larger levels of anisotropy is not clear, beyond the fact that ψ evidently should not change sign. We shall consequently suggest a rather conservative interpolation formula, devoid of extreme behaviour.

We can write

$$\psi = \tfrac{14}{5} + (3\cdot78 - \tfrac{14}{5})\, G(\mathrm{II}) \exp(-ER_l^{-\frac{1}{2}}), \tag{38}$$

where $G(0) = 1$ and $G(\frac{2}{3}) = 0$. If we write, for small II, $G(\mathrm{II}) = 1 - c\mathrm{II}$, then for large Reynolds number and small anisotropy we have

$$\psi \simeq 3\cdot78 + (3\cdot78 - \tfrac{14}{5})(-ER_l^{-\frac{1}{2}} - c\mathrm{II}) + \dots . \tag{39}$$

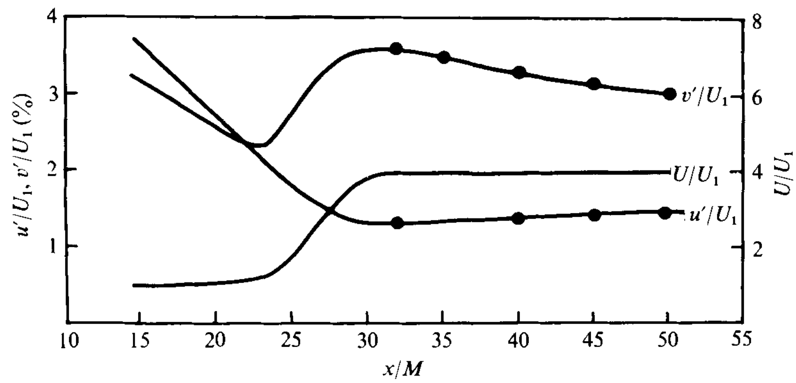

FIGURE 6. Solid lines are those of Uberoi (1956, figure 5). Points are our predictions. $R_M = U_1 M/\nu = 3710$, $M = 2$ in., contraction $= 4:1$.

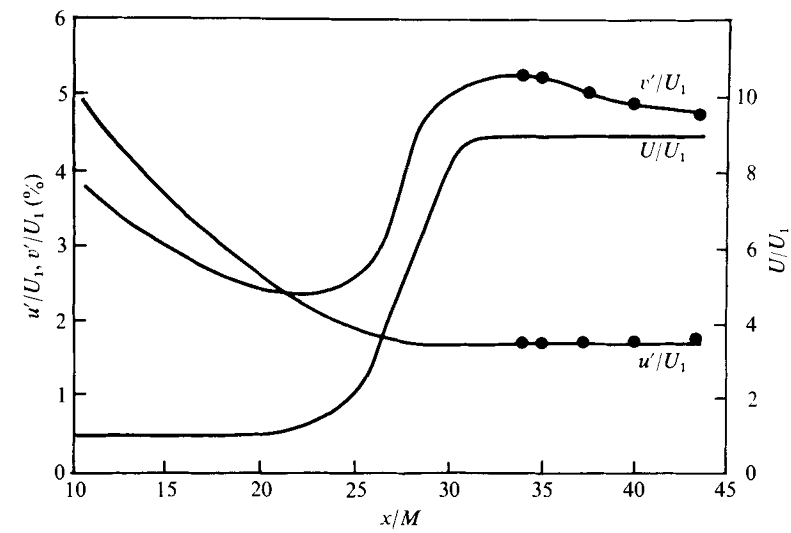

FIGURE 7. Solid lines are those of Uberoi (1956, figure 6). Points are our predictions. $R_M = U_1 M/\nu = 3710$, $M = 2$ in., contraction $= 9:1$.

If we require agreement with (36), we must have $E = 2{\cdot}83$ and $c = 9{\cdot}28$. The form of $G(\mathrm{II})$ presents a problem owing to our lack of information, and we cannot do much more than take a smooth, monotone form like $1 - 0{\cdot}337 \ln (1 + 27.5\,\mathrm{II})$, which goes to zero at $\frac{2}{3}$ and has the required slope and value at the origin. Hence the final form suggested is

$$\psi = \tfrac{14}{5} + 0{\cdot}980 \exp(-2{\cdot}83 R_l^{-\frac{1}{2}})\,[1 - 0{\cdot}337 \ln (1 + 27{\cdot}5\,\mathrm{II})]. \tag{40}$$

6. Results and discussion

From (34) and (40), all the flows of Uberoi (1956, 1957) and of Mills & Corrsin (1959) were predicted using a simple forward-difference scheme and taking values after the contraction as the initial conditions. Values of ϵ were selected to reproduce the initial

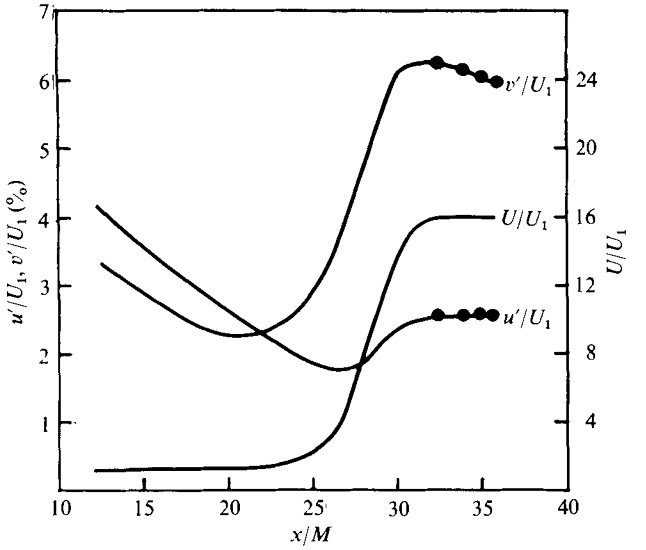

FIGURE 8. Solid lines are those of Uberoi (1956, figure 7). Points are our predictions. $R_M = U_1 M/\nu = 3710$, $M = 2$ in., contraction $= 16:1$.

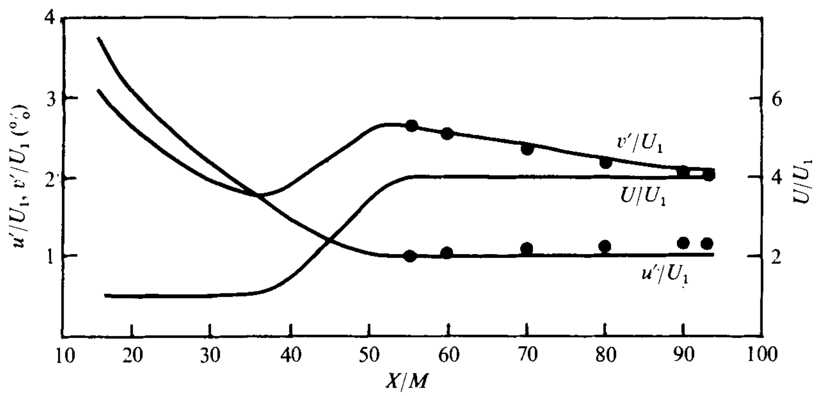

FIGURE 9. Solid lines are those of Uberoi (1956, figure 10). Points are our predictions. $R_M = U_1 M/\nu = 6150$, $M = 1$ in., contraction $= 4:1$.

slope of the energy. The results are shown in figures 6–12. The continuous lines are the curves faired through the data by the authors, while the points are our calculations. Note that the flows of figures 6, 9, 10 and 12 had been used to optimize values of a and D. However, the ability to reproduce the shape of the curves, and to reproduce all the curves without regard to the Reynolds number, is still significant. The predictions are seen to be essentially within experimental accuracy, with the exception of the flow of Uberoi (1956). These measurements are somewhat pathological, in the sense that they are the only measurements ever presented which show the streamwise and cross-stream energies becoming equal after a finite extent of duct. From a number of points of view this appears to be unlikely, so that it is in a sense encouraging that our calculations indicate a finite anisotropy at the end of the duct.

Reynolds (1974) has suggested that ϕ_{ij} cannot depend on b_{ij}^2 because of the anisotropy

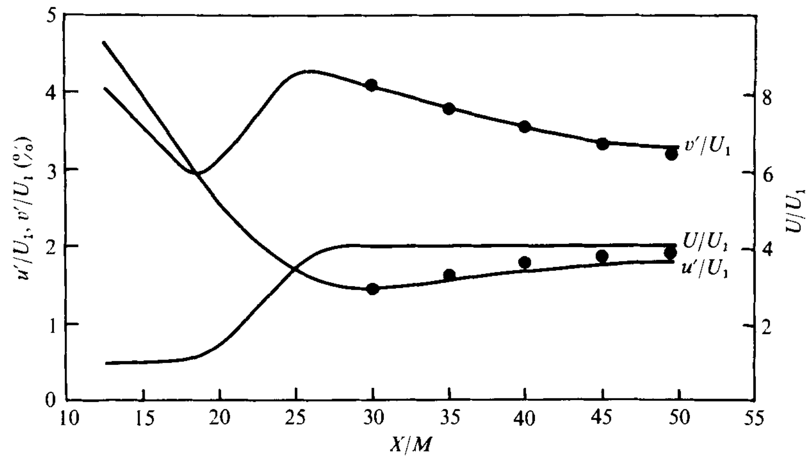

FIGURE 10. Solid lines are those of Uberoi (1956, figure 11). Points are our predictions. $R_M = U_1 M/\nu = 12300$, $M = 2$ in., contraction = 4:1.

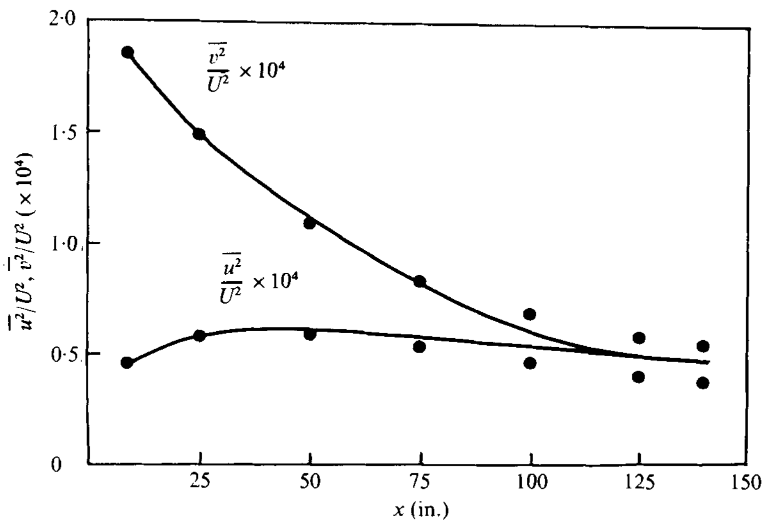

FIGURE 11. Solid lines are those of Uberoi (1957, figure 2). Points are our predictions. $R_M = U_1 M/\nu = 10000$, $M = 2$ in., contraction = 4:1.

in response this would induce. That is, considering axisymmetric turbulence, we have found that ϕ_{11}/b is not symmetric for (moderate) positive and negative values of b; for small values of b, ϕ_{11}/b is symmetric, from (19). It can be seen, however, that for moderate values of b it cannot be symmetric, since it must vanish for $b = -\frac{1}{3}$ and for $b = +\frac{2}{3}$. Turbulence which is almost one-dimensional is dynamically quite different from turbulence which is almost two-dimensional, and there is no reason to expect symmetry.

In the same work, Reynolds (1974) has presented an argument equivalent to the statement that ϕ_{ij} cannot be simply a function of b_{ij}, since b_{ij} is even in u_i whereas $\overline{p_{,i} u_j}$ must be odd in u_i. The remark is not correct, however. Inspection of the Navier–Stokes equations shows that reflexion of the co-ordinate system (which is the only

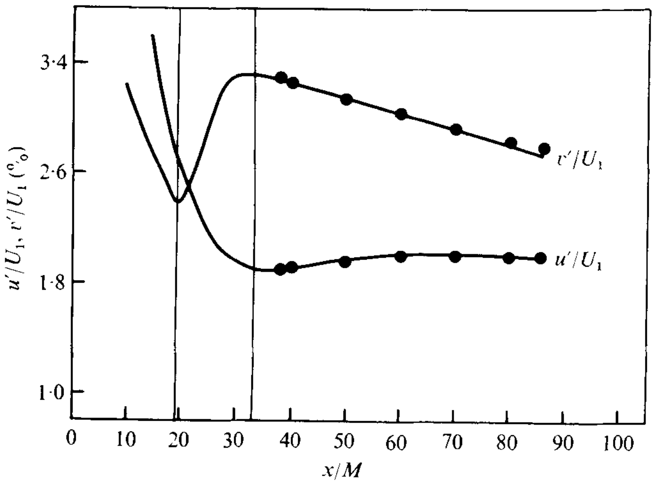

FIGURE 12. Solid lines are those of Mills & Corrsin (1959, figure 5*b*). Points are our predictions. $R_M = U_1 M/\nu = 7420$, $M = 1$ in., contraction = 4:1.

correct way to effect a reversal of the velocity field) reverses $p_{,i}$ (as it does also $u_{i,j}u_j$); that is, the direction of the gradient must reverse also in mirror turbulence. Thus $\overline{p_{,i}u_j}$ does not reverse, and hence this does not provide an argument against its being represented as a function of b_{ij}.

Schumann & Patterson (1977) calculate values of γ, assuming it to be constant, and find that it must change sign with the sign of III. This is consistent with our finding that γ_0 must vanish at infinite Reynolds number. Although γ is not antisymmetric, for the reasons stated above, we may expect that the symmetric part will be relatively small.

This work has evolved over a considerable period and has benefited from the criticism of many persons, particularly our colleagues at The Pennsylvania State University. The investigation began as the result of a discussion with J. Wyngaard, for which we are grateful. The work was supported in part by the U.S. National Science Foundation, Atmospheric Sciences Section, under Grant no. ATM75-13357 A01, and in part by the U.S. Environmental Protection Agency through its Select Research Group in Air Pollution Meteorology.

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